Section 2.3. LINEAR EQUATIONS

Defn: A first-order ODE of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x) \qquad (1)$$

is said to be a **linear equation** in the variable y.

By dividing both sides of (1) by $a_1(x)$, we get a more useful form, the **standard form**, of the linear equation

$$\frac{dy}{dx} + P(x)\,y = f(x) \qquad (2).$$

for example:

$$\frac{dy}{dx} + 2xy = 0$$

and

$$\frac{dy}{dx} = y + 5 \qquad \text{are linear and separable.}$$

However

$$\frac{dy}{dx} + y = x \qquad \text{is linear, but non separable.}$$

METHOD OF SOLUTION.

We want to transform the LHS of (2) into the derivative of a product by multiplying both sides of (2) by a special function $\mu(x)$:

$$\frac{dy}{dx}[\mu(x)\, y] = \text{LHS} \times \mu(x)$$

$$= \left(\frac{dy}{dx} + P(x)\right)\mu(x)$$

Assume we can find $\mu(x)$ as above, then

$$\frac{dy}{dx}[\mu(x)\, y] = \mu(x) f(x) \quad (\text{LHS} = \text{RHS})$$

$$\Rightarrow \quad \mu(x)\, y = \int \mu(x) f(x)$$

$$\Rightarrow \quad y = \mu(x)^{-1} \int \mu(x) f(x) + C\,\mu(x)^{-1}$$

is a solution.

How to find such function $\mu(x)$?

$$\frac{d}{dx}[\mu(x)y] = LHS \times \mu(x)$$

$$\mu \frac{dy}{dx} + \frac{d\mu}{dx} y = \mu \frac{dy}{dx} + \mu P y$$

We want to have

$$\frac{d\mu}{dx} = \mu P$$

$$\Leftrightarrow \quad \frac{d\mu}{\mu} = P\,dx$$

$$\Leftrightarrow \quad \ln|\mu(x)| = \int P(x)\,dx + C_1$$

$$\to \quad \mu = C_2 e^{\int P(x)dx}$$

The function $\mu(x) = e^{\int P(x)\,dx}$ is called the **integrating factor** of eq. (2).

<u>Example:</u> Solve $\frac{dy}{dx} - 3y = 0$

We have $P(x) = -3$, thus the integrating factor is

$$\mu(x) = e^{\int(-3)\,dx} = e^{-3x}.$$

Multiply both sides by $\mu(x)$ we get

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x} y = e^{-3x} \cdot 0$$

$$\Leftrightarrow \quad \frac{d}{dx}\left[e^{-3x}y\right] = 0$$

$$\Leftrightarrow \quad e^{-3x}y = c \quad \text{or} \quad y = c.e^{3x}. \qquad \square$$

<u>Example 2:</u> Solve $\frac{dy}{dx} - 3y = 6$

$$\Leftrightarrow \quad \frac{d}{dx}\left[e^{-3x}y\right] = 6.e^{-3x}$$

$$\Leftrightarrow \quad e^{-3x}y = 6\int e^{-3x}\,dx$$

$$\Leftrightarrow \quad e^{-3x}y = 6\left(\frac{e^{-3x}}{-3}\right) + c$$

$$\boxed{y = -2 + c\,e^{3x}}. \qquad \square$$

Note that if P and f are continuous in an interval I, then

$$y = \mu^{-1}\int \mu f\,dx + c\,\mu^{-1}$$

$$(4) \quad y = e^{-\int p\,dx}\int e^{\int p\,dx} f(x)\,dx + c.e^{-\int p\,dx}$$

is a solution of (2), and any solution of (2) defined on I is a member of this family. We call (4) the **general solution** of the equation on I.

In this case the interval of existence and uniqueness I_0 is the entire the interval I.

<u>Example 3:</u> Find the general solution

$$x \frac{dy}{dx} - 4y = x^6 e^x$$

$$\Leftrightarrow \quad \frac{dy}{dx} - \frac{4}{x} y = x^5 e^x \qquad (7)$$

$$\Rightarrow \quad P(x) = -\frac{4}{x}, \quad f(x) = x^5 e^x$$

We have P and f are continuous on $I = (0, \infty)$.
Hence the integrating factor

$$\mu(x) = e^{\int P(x)\,dx} = e^{-4\int \frac{1}{x}dx}$$

$$= e^{-4 \ln x} \quad \left(\ln x = \ln|x| \text{ since } x > 0 \right)$$

$$= x^{-4}$$

Thus multiply both sides of (7) by $\mu(x) = x^{-4}$

$$x^{-4} \frac{dy}{dx} - \frac{4}{x^5} y = x e^x$$

$$\frac{d}{dx}\left[x^{-4} y\right] = x e^x$$

$$x^{-4} y = \int x e^x \, dx$$

$$= x e^x - e^x + C$$

$$\Leftrightarrow \quad y = x^5 e^x - x^4 e^x + C x^4$$

is the general solution defined on $(0, \infty)$.

Example 4. Find the general solution

$$(x^2-9)\frac{dy}{dx} + xy = 0$$

$$\Leftrightarrow \quad \frac{dy}{dx} + \frac{x}{x^2-9}\, y = 0$$

$$P(x) = \frac{x}{x^2-9}, \quad f(x) = 0$$

We have P and f are continuous on $(-\infty,-3)$, $(-3,3)$, and $(3,\infty)$.

$\boxed{\text{Solve on } (3,\infty):}$

$$\mu(x) = e^{\int \frac{x}{x^2-9}dx} = e^{\frac{1}{2}\int \frac{dx^2}{x^2-9}}$$

$$= e^{\frac{1}{2}\ln|x^2-9|}$$

$$= \sqrt{x^2-9} \quad (x^2>9 \text{ as } x>3).$$

Multiply both sides of the equation by $\mu(x)$:

$$\sqrt{x^2-9}\,\frac{dy}{dx} + \frac{x\sqrt{x^2-9}}{x^2-9}\, y = 0$$

$$\Leftrightarrow \quad \frac{d}{dx}\left[\sqrt{x^2-9}\; y\right] = 0$$

$$\Leftrightarrow \quad \sqrt{x^2-9}\; y = c \quad \Leftrightarrow \quad y = \frac{c}{\sqrt{x^2-9}}.$$

We have the same solution on $(-\infty,-3)$.

However, $x \in (-3,3)$ we get $y = \dfrac{c}{\sqrt{9-x^2}}$.

Example: Solve an IVP

$$\frac{dy}{dx} + y = x \quad , \ y(0) = 4$$

$$\mu(x) = e^{\int dx} = e^x \quad (P = 1, f(x) = x).$$

$$\Rightarrow e^x \frac{dy}{dx} + e^x y = x e^x$$

$$\Leftrightarrow \frac{d}{dx}(e^x y) = \int x e^x$$

$$\Leftrightarrow e^x y = x e^x - e^x + c$$

$$\Leftrightarrow y = x - 1 + c/e^x.$$

Substitute $x = 0$, $y = 0$, we get $c = 5$.

$$\Rightarrow y = x - 1 + 5e^{-x}.$$

PIECEWISE-LINEAR D.E.

$$\frac{dy}{dx} + P(x)\, y = f(x)$$

Where $f(x)$ is piecewise-defined function.

Example 6: Solve $\frac{dy}{dx} + y = f(x)$, $y(0) = 0$

where $f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & x > 1. \end{cases}$

Consider $0 \le x \le 1$. Our eq becomes

$$\frac{dy}{dx} + y = 1$$

$\left(P(x) = 1, \text{ so } \mu(x) = e^{\int p\,dx} = e^x \right)$

$$\Leftrightarrow \quad \frac{d}{dx}[e^x y] = e^x$$

$$\Leftrightarrow \quad e^x y = e^x + c$$

$$\Leftrightarrow \quad y = 1 + c \cdot e^{-x}.$$

Since the initial point $x_0 = 0$ in this interval,

$$0 = 1 + c \cdot e^{-0} \Leftrightarrow c = -1.$$

$\Rightarrow$ solution is $y = 1 - e^{-x}$.

Consider $x > 1$. Our eq is

$$\frac{dy}{dx} + y = 0$$

$$\Leftrightarrow \quad \frac{d}{dx}[e^x y] = 0$$

$$\Leftrightarrow \quad e^x y = c$$

$$\Leftrightarrow \quad y = c \cdot e^{-x}$$

Thus our solution is

$$y(x) = \begin{cases} 1 - e^{-x} & 0 \le x \le 1; \\ c \cdot e^{-x} & x > 1. \end{cases}$$

Reading exercise: ERROR FUNCTION. (pages 60-61)